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DIFFERENCE APPROXIMATIONS TO INTERFACE CURVES FOR NONLINEAR DIFFUSION EQUATIONS WITH ABSORPTION

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Abstract—We consider some nonlinear reaction-diffusion equations with extinction phenomena in finite time. For the solution v, there appear *interface curves* between $v > 0$ and $v = 0$. We propose difference approximations to interface curves, and prove the convergence to exact interface curves.

INTRODUCTION

We are concerned with difference approximations to the following initial value problem for the nonlinear diffusion equation described by

$$v_t = (v^m)_{xx} - cv^p, \qquad x \in \mathbf{R}^1, \quad t > 0 \tag{1}$$

with an initial condition

$$v(0, x) = v^0(x), \qquad x \in \mathbf{R}^1, \tag{2}$$

where $m(> 1)$, $p(> 0)$ and $c(\geq 0)$ are constants, and $v^0(x)$ has compact support. The equation of the form (1) is known as a simple mathematical model for several physical phenomena.

The first model, with $c = 0$, describes the flow of an ideal gas through a homogeneous porous medium, where v represents a density of the gas. Physically, v^{m-1} is the pressure of the gas and $(v^{m-1})_x$ is the velocity.

The second model, with $c > 0$, describes the transport of the thermal energy in plasma. Here v means the temperature. The term $-cv^p$ is understood as volumetric absorption caused by radiation.

In both models, with $c = 0$ and with $c > 0$, the most interesting phenomenon is the occurrence of finite propagation of the initial support. It is already shown that there are three cases of the behavior of supp $v(t, \cdot)$.

Case 1. Positivity. For $c = 0$ and $m > 1$, or $c > 0$ and $p \geq m > 1$ supp $v(t, \cdot)$ expands as t increases and $\lim_{t \to \infty}$ supp $v(t, \cdot) = \mathbf{R}^1$ [1-6].

Case 2. Localization. For $c > 0$ and $m > p \geq 1$ supp $v(t, \cdot)$ expands as t increases and is uniformly bounded with respect to t [5,7-9].

Case 3. Total Extinction. For $c > 0$, $m > 1$ and $0 < p < 1$ supp v is compact in $[0, \infty) \times \mathbf{R}^1$ and $v(t, x)$ extincts in finite time [5,8,10].

From a numerical point of view, it is interesting to determine the behavior of supp $v(t, \cdot)$, that is, *interface curves* appearing between $v > 0$ and $v = 0$.

Several difference schemes to (1) and (2) with $c = 0$ have been investigated. In [11], Graveleau and Jamet proved the finite propagation of the support by using their difference scheme. However,

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX their scheme does not give good approximations to the exact interface curves. DiBenedetto and Hoff's scheme [12] and Mimura, Nakaki and Tomoeda's scheme [13] give the convergence of numerical interface curves. It is observed that the numerical interface curves by the latter scheme are more accurate than the former's.

Numerical computations to (1) and (2) with $c > 0$ are investigated by Rosenau and Kamin [14], Mimura, Nakaki and Tomoeda [13] and Nakaki [15]. In [13], the convergence of numerical solutions is proved for Cases 1 and 2, and the convergence of numerical interface curves is also proved for Case 1 and Case 2 with $p = 1$. In Case 3, Rosenau and Kamin numerically examined the problem of the pulse splitting into several sub-pulses, but the theoretical results of the numerical scheme are not discussed. In Case 3, with $m+p = 2$, it is shown in [15] that not only numerical solutions, but also interface curves converge to exact ones under

Condition A: $(v^0(x))^{m-1}$ is concave downward on its support.

In this paper, we show the convergence of numerical interface curves without Condition A for Case 3 with $m + p = 2$. However, instead of Condition A, we have to impose the following condition on numerical results.

Condition B: There exist positive numbers M, T^* and h^* such that

$$\dot{\ell}_h(t) < M \qquad \text{and} \qquad \dot{r}_h(t) > -M \quad \left(\dot{} = \frac{d}{dt}\right) \tag{3}$$

for almost all $t \in [0, T^*]$ and for all $h \in (0, h^*)$. Here $\ell_h(t)$ and $r_h(t)$ denote left and right numerical interface curves, respectively, and h is a space mesh width.

In the following, we consider the difference approximations to (1) and (2) in Case 3 with $m + p = 2$.

DIFFERENCE SCHEMES

To show the stability and convergence, we set $u = v^{m-1}$ and rewrite (1) and (2) as

$$u_t = m\,u\,u_{xx} + a(u_x)^2 - c', \qquad a = \frac{m}{m-1}, \quad c' = (m-1)\,c, \tag{4}$$

$$u(0,x) = u^0(x) \equiv (v^0(x))^{m-1}. \tag{5}$$

Our difference scheme approximates the problems (4) and (5) instead of (1) and (2), and is constructed based on splitting the equation (4) into three parts:

$$u_t = Du = -c', \tag{6}$$

$$u_t = Hu = a(u_x)^2, \tag{7}$$

$$u_t = Pu = m\,u\,u_{xx}. \tag{8}$$

Then our difference scheme [15] is described as follows:

Find the sequence $\{u_h^n\}_{n=0,1,2,\dots} \subset V_h$ such that

$$u_h^{n+1} = S_{h,k}\,u_h^n \equiv (P_{h,k/\mu})^{\mu} \cdot H_{h,k} \cdot (D_{h,k})\,u_h^n \qquad \text{for } n = 0,1,2,\dots. \tag{9}$$

Here $k \equiv k_{n+1} \equiv t_{n+1} - t_n$ is a variable time step, $\mu \equiv \mu_{n+1}$ is an integer depending on k_{n+1}, and V_h is the set of the non-negative continuous functions u_h with the following properties:

(i) u_h has compact support $[\ell(u_h), r(u_h)]$;

(ii) u_h is linear on each interval $[x_i, x_{i+1}]$ $(i \in \mathbb{Z})$, where

$$x_i = ih \qquad \text{for all } ih \in (\ell(u_h), r(u_h)) \quad (i = L, L+1, \dots, R-1, R),$$
$$x_{L-1} \equiv \ell(u_h), \quad x_{R+1} \equiv r(u_h).$$

The solution of (1) and (2) extincts in finite time, and the front of support may expand and/or shrink. Taking these properties into consideration, we construct difference operators $D_{h,k}$, $H_{h,k}$ and $P_{h,k/\mu}$ mapping from V_h to V_h, which are approximating (6), (7) and (8), respectively. Let $h_i = x_{i+1} - x_i$ and $u_i = u_h(x_i)$.

Difference operator $D_{h,k}$

$$\ell(D_{h,k}\,u_h) = \sup\{\xi \mid u_h(x) - c'k \le 0 \quad \text{on } (-\infty,\xi]\}, \tag{10}$$
$$r(D_{h,k}\,u_h) = \inf\{\xi \mid u_h(x) - c'k \le 0 \quad \text{on } [\xi,\infty)\}, \tag{11}$$
$$(D_{h,k}\,u_h)(x_i) = \{[u_h - c'k]^+\}(x_i) \quad \text{for all } i \in \mathbf{Z}, \tag{12}$$

where $[f]^+ = \max\{f, 0\}$.

Difference operator $H_{h,k}$

$$\ell(H_{h,k}u_h) = \ell(u_h) - a\delta u_{L-1}k, \tag{13}$$
$$r(H_{h,k}u_h) = r(u_h) - a\delta u_R k, \tag{14}$$

$$(H_{h,k}u_h)(x_i) = \text{exact solution } u(k,x_i) \text{ of } u_t = Hu \quad \text{with the initial value } u(0,x) = u_h(x). \tag{15}$$

Let

$$L' = \min\{i \mid ih \in (\ell(H_{h,k}\,u_h), r(H_{h,k}\,u_h))\}, \qquad R' = \max\{i \mid ih \in (\ell(H_{h,k}\,u_h), r(H_{h,k}\,u_h))\}.$$

Then it follows that

$$H_{h,k}u_h = \begin{cases} u_i + a(\delta u_i)^2 k & \text{if } i \in S^+ = S_S^+ \cup S_R^+, \\ u_i + a(\delta u_{i-1})^2 k & \text{if } i \in S^- = S_S^- \cup S_R^-, \\ u_i & \text{if } i \in S^0, \\ (L'h - \ell')\,\delta u_{L-1} & \text{if } i = L' = L-1, \\ (R'h - r')\,\delta u_R & \text{if } i = R' = R+1, \\ 0 & \text{if } i \in \mathbf{Z} \setminus \{L', \dots, R'\}, \end{cases} \tag{16}$$

where

$$\begin{aligned} S_S^+ &= \{i \in \{L,\dots,R\} \mid \delta u_{i-1} < \delta u_i \quad \text{and} \quad \delta u_{i-1} > -\delta u_i\}, \\ S_S^- &= \{i \in \{L,\dots,R\} \mid \delta u_{i-1} < \delta u_i \quad \text{and} \quad \delta u_{i-1} \le -\delta u_i\}, \\ S_R^+ &= \{i \in \{L,\dots,R\} \mid \delta u_{i-1} \ge \delta u_i > 0\}, \\ S_R^- &= \{i \in \{L,\dots,R\} \mid 0 > \delta u_{i-1} \ge \delta u_i\}, \\ S^0 &= \{i \in \{L,\dots,R\} \mid \delta u_{i-1} \ge 0 \ge \delta u_i\}, \\ \delta u_i &= \frac{u_{i+1} - u_i}{h_i}. \end{aligned}$$

Stability Condition: $k \equiv k_{n+1}$ is the largest number satisfying the following inequalities:

$$a\|(u_h)_x\|_\infty\, k \le \min\left\{\frac{h}{4},\ Lh - \ell(u_h),\ r(u_h) - Rh\right\}, \tag{17}$$
$$k \le Ch^s \qquad \text{(For simplicity we put } C = 1,\ s = 1/2). \tag{18}$$

Difference operator $P_{h,k/\mu}$

$$\begin{aligned} (P_{h,k/\mu}\,u_h)(x_i) &= u_i + \frac{k}{\mu}\, m\, u_i\, \delta^2 u_i \qquad \text{for all } i \in \mathbf{Z}, \\ \delta^2 u_i &= 2\,\frac{\delta u_i - \delta u_{i-1}}{h_i + h_{i-1}}. \end{aligned} \tag{19}$$

Stability Condition: μ satisfies the following inequalities:

$$m\|u_h\|_\infty (k/\mu)\left[\frac{1}{h^2} + \frac{2}{h(h+h_j)}\right] \le 1 \qquad \text{for } j = L-1, R, \tag{20}$$
$$4m\|(u_h)_x\|_\infty\, \frac{k/\mu}{h+h_j} \le 1 \qquad \text{for } j = L-1, R. \tag{21}$$

To start the scheme (9), we take

$$t_0 = 0, \quad \ell_0 = \ell(u_h^0), \quad r_0 = r(u_h^0), \quad u_h^0(x_i) = u^0(x_i). \tag{22}$$

When $D_{h,k}\,u_h^n \equiv 0$ for $k = h/(4a\|(u_h^n)_x\|_\infty)$, we denote numerical extinction time by $T_h^* = t_n + \|u_h^n\|_\infty/c'$, and stop the numerical computation. Otherwise, we take k_{n+1} the largest number satisfying

(i) (17) and (18) with $u_h = D_{h,k}\,u_h^n$;
(ii) Every connected components of the set [supp u_h^n] \ [supp $D_{h,k}\,u_h^n$] has at most one point x such that x/h is an integer.

STABILITY AND CONVERGENCE

We introduce the following condition.
Condition C. $u^0 \in C^0(\mathbf{R}^1) \cap BV(\mathbf{R}^1)$ is a non-negative function with compact support $[\ell(u^0), r(u^0)]$ and satisfies $u_x^0 \in L^\infty(\mathbf{R}^1) \cap BV(\mathbf{R}^1)$ and

$$u_{xx}^0(x) > -C_1 \qquad \text{for some positive constant } C_1. \tag{23}$$

THEOREM 1 (STABILITY). *Assume Condition C. Then*

$$0 \le u_h^n \le \|u^0\|_\infty \qquad \textit{for all } n \ge 0. \tag{24}$$

Moreover, the following estimates

$$(u_h^n)_{xx} > -C_1, \tag{25}$$

$$\|(u_h^n)_x\|_\infty \le \|(u_h^0)_x\|_\infty, \quad \|(u_h^n)_x\|_{L^1(\mathbf{R}^1)} \le \|(u_h^0)_x\|_{L^1(\mathbf{R}^1)}, \quad V((u_h^n)_x) \le V((u_h^0)_x), \tag{26}$$

$$\ell_0 - a\|u_x^0\|_\infty t_n \le \ell_n \le r_n \le r_0 + a\|u_x^0\|_\infty t_n, \tag{27}$$

$$\left\| \frac{u_h^{n+1} - u_h^n}{k_{n+1}} \right\|_{L^1(\mathbf{R}^1)} \le m\|u^0\|_\infty\, V(u_x^0) + a\|u_x^0\|_\infty\, \|u_x^0\|_{L^1(\mathbf{R}^1)} + c'(r_0 - \ell_0 + 2a\,\|u_x^0\|_\infty\, t_n) \tag{28}$$

hold for all $n \ge 0$, where $V(f)$ denotes the total variation of f on $\mathbf{R}^1$.

PROOF. It is proved in [13,16] that numerical solutions u_h^n computed by the scheme (9) to the equation (4) with $c = 0$ satisfy (24)–(28). For $u_h' = D_{h,k}u_h$ with $c > 0$, we can easily show that

$$0 \le u_h' \le \|u_h\|_\infty, \qquad (u_h')_{xx} \ge (u_h)_{xx},$$

$$\|(u_h')_x\|_\infty \le \|(u_h)_x\|_\infty, \quad \|(u_h')_x\|_{L^1(\mathbf{R}^1)} \le \|(u_h)_x\|_{L^1(\mathbf{R}^1)}, \quad V((u_h')_x) \le V((u_h)_x),$$

which yield (24)–(26). Since $\ell(u_h) < \ell(u_h')$ and $r(u_h') < r(u_h)$ hold by (10) and (11), we have (27). From (12) it follows that

$$\left\| \frac{u_h' - u_h}{k} \right\|_{L^1(\mathbf{R}^1)} \le c'(r(u_h) - \ell(u_h)).$$

Thus (28) holds by (27), and the proof is complete.

Under Condition B, we define a function $u_h(t,x)$ by

$$u_h(t,x) = u_h^n(x) \quad \text{on } [t_n, t_{n+1}) \times \mathbf{R}^1 \qquad \text{for all } t_n \le T^* \quad \text{and } h < h^*.$$

We put $\ell_n = \ell(u_h^n)$ and $r_n = r(u_h^n)$, and define the left (respectively, right) numerical interface curve $\ell_h(t)$ (respectively, $r_h(t)$) by piecewise-linearly interpolating (t_n, ℓ_n) (respectively, (t_n, r_n)) $(n \ge 0)$.

Theorem 2 (Convergence). *Assume Conditions B and C. Then there exist Lipschitz continuous functions $\ell^*(t)$ and $r^*(t)$ on $[0,T^*)$ such that*

$$\|v_h - v\|_{L^\infty(H)} \longrightarrow 0 \qquad \text{as } h \to 0, \tag{29}$$

$$\|\ell_h - \ell^*\|_{L^\infty([0,T^*))} \longrightarrow 0 \qquad \text{as } h \to 0, \tag{30}$$

$$\|r_h - r^*\|_{L^\infty([0,T^*))} \longrightarrow 0 \qquad \text{as } h \to 0, \tag{31}$$

where $H = [0,T^*) \times \mathbf{R}^1$, $v_h \equiv (u_h)^{1/(m-1)}$ and $v(t,x)$ is the unique weak solution of (1) and (2). Moreover, ℓ^* and r^* become the left and right interface curves, respectively.

Proof. By using Theorem 6.1 in Graveleau and Jamet [11] and the estimates (24), (26) and (28), we can choose a subsequence $\{u_{h'}\}$ which uniformly converges to a weak solution u of (4). This implies that $(u_{h'})^{1/(m-1)}$ also uniformly converges to a weak solution of (1). Thus (29) holds by the uniqueness [4,5] of the solution of (1) and (2).

Next we show (30). From (27) it follows that

$$\ell_0 - a\|(u_h^0)_x\|_\infty T^* \le \ell_h(t) \le r_0 + a\|(u_h^0)_x\|_\infty T^* \qquad \text{on } [0,T^*]. \tag{32}$$

By Condition B and (32) we can apply Ascoli-Arzela's Theorem to the sequence $\{\ell_h\}$. Thus, there exists a Lipschitz continuous function $\tilde{\ell}(t)$ and subsequence $\{\ell_{h'}\}$ satisfying

$$\|\ell_{h'} - \tilde{\ell}\|_{L^\infty([0,T^*])} \longrightarrow 0 \qquad \text{as } h' \to 0.$$

Now we show that $\tilde{\ell}(t)$ is the left interface curve of the weak solution v of (1) and (2). For each fixed $t^* \in [0,T^*)$, there exist integers n and L such that $t_n \le t^* < t_{n+1}$ and $(L-1)h \le \ell_h(t_n) < Lh$. Then for any integer $p > 0$ we have

$$\begin{aligned} u_h(t^*, (L+p)\,h) &\ge h \sum_{i=L}^{L+p-1} \delta u_i^n = h \sum_{j=0}^{p-1} \left\{ \delta u_{L-1}^n + \sum_{i=L}^{L+j} (\delta u_i^n - \delta u_{i-1}^n) \right\} \\ &\ge p\,h\,\delta u_{L-1}^n - h \sum_{j=0}^{p-1} (j+1)\,h\,C_1 \ge p\,h\,\delta u_{L-1}^n - C_1\,h^2\,\frac{p(p+1)}{2}, \end{aligned} \tag{33}$$

where we use h instead of h'. Similarly,

$$u_h(t^*, (L+p)\,h) \ge h\,\delta u_L^n + h \sum_{j=1}^{p-1} \left\{ \delta u_L^n + \sum_{i=L+1}^{L+j} (\delta u_i^n - \delta u_{i-1}^n) \right\} \ge p\,h\,\delta u_L^n - C_1\,h^2\,\frac{p(p-1)}{2}. \tag{34}$$

On the other hand, by Condition B we have

$$\ell_{n+1} - \ell_n = \begin{cases} \dfrac{c' k_{n+1}}{\delta u_{L-1}^n} - a\,\delta u_{L-1}^n\,k_{n+1} \le M k_{n+1} & \text{if } c' k_{n+1} < u_L^n, \\[2ex] \dfrac{u_L^n}{\delta u_{L-1}^n} + \dfrac{c' k_{n+1} - u_L^n}{\delta u_L^n} - a\,\delta u_L^n\,k_{n+1} \le M k_{n+1} & \text{if } c' k_{n+1} \ge u_L^n \quad \text{and } \delta u_L^n \ne 0, \\[2ex] \dfrac{u_L^n}{\delta u_{L-1}^n} + h - a\,\delta u_{L+1}^n\,k_{n+1} \le M k_{n+1} & \text{if } c' k_{n+1} = u_L^n \quad \text{and } \delta u_L^n = 0. \end{cases}$$

From this it follows that

$$\frac{c' k_{n+1}}{\max\{\delta u_{L-1}^n, \delta u_L^n\}} \le (M + a\|u_x^0\|_\infty)\, k_{n+1},$$

which yields

$$\max\{\delta u_{L-1}^n, \delta u_L^n\} \ge \varepsilon \equiv \frac{c'}{M + a\|u_x^0\|_\infty} > 0. \tag{35}$$

We put $\eta = ph$ and fix it. By (33)–(35) we have

$$u_h(t^*, (L+p)\,h) \ge \eta \left\{ \varepsilon - \frac{C_1(\eta + h)}{2} \right\}.$$

Letting $h \to 0$, we obtain

$$u(t^*, \tilde{\ell}(t^*) + \eta) \ge \eta \left\{ \varepsilon - \frac{C_1 \eta}{2} \right\}.$$

Hence,

$$u(t^*, \tilde{\ell}(t^*) + \eta) \ge \frac{\eta \varepsilon}{2} > 0 \qquad \text{for } \eta < \frac{\varepsilon}{C_1},$$

which implies that $\tilde{\ell}(t) = \ell^*(t)$ on $[0, T^*)$. Thus (30) holds. Similarly, (31) can be shown, and the proof is complete.

NUMERICAL EXPERIMENTS

We carry on a numerical computation with $m = 1.5$, $p = 0.5$ and $c = 1$ for the following initial function.

$$v^0(x) = \begin{cases} 0.75(x^2-1)^2 - 4(x^2-1)(x^2-0.125) & \text{for } x^2 \le 1, \\ 0 & \text{for } x^2 > 1. \end{cases}$$

In the following table we observe that Condition B seems to be satisfied, and that the numerical interface curve converges as $h \to 0$.

Numerical extinction time T_h^* and $T_M = \sup\{T \mid \dot{\ell}_h(t) < M,\ \dot{r}_h(t) > -M \text{ on } [0,T)\}$ (see Condition B) with $M = 50, 25$, and numerical right interface curve $r_h(t)$ at $t = 0.5, 1$.

h	T_h^*	T_{50}	T_{25}	$r_h(0.5)$	$r_h(1)$
2^{-3}	1.234850	1.228411	1.220418	1.999660	1.437322
2^{-4}	1.258987	1.255664	1.253092	2.071109	1.544311
2^{-5}	1.270498	1.268291	1.264852	2.107836	1.594681
2^{-6}	1.276316	1.274935	1.271461	2.127088	1.620654
2^{-7}	1.279211	1.277847	1.274409	2.136811	1.633572
2^{-8}	1.280641	1.279448	1.275864	2.141667	1.639971
2^{-9}	1.281343	1.280155	1.276659	2.144063	1.643136

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